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ON THE SUBSONIC STATIONARY MOTION OF STAMPS AND FLEXIBLE COVER-PLATES ON THE BOUNDARY OF AN ELASTIC HALF-PLANE AND A COMPOSITE PLANE*

E.L. NAKHMEIN and B.M. NULLER

A mixed dynamic problem for an elastic half-plane on different sections of whose boundary shear and normal stresses and displacements are given simultaneously in four fundamental combinations is considered. It is assumed that all the sections move at an identical constant subsonic velocity along the half-plane boundary and their number and mutual arrangement are arbitrary. An analogous problem on the interaction of two half-planes of different materials (a composite plane) is examined under the formulation of six kinds of contact conditions simultaneously in two modifications. The solutions are constructed in quadratures on the basis of new representations of the complex Galin potentials /1/. The first problem is reduced to a scalar combined Hilbert-Riemann boundary value problem /2/ for a plane with slots, and the second to unrelated Hilbert-Riemann and Hilbert problems for the same domain. Both problems of the theory of analytic functions are solved by a new method different from /2/. The problem of the wedging of a composite plane by a finite stamp moving at a sub-Rayleigh velocity /3/, and the problem of the motion of a stamp and a flexible cover plate over a half-plane boundary at subsonic velocity are examined as examples.

The exact solutions of stationary contact problems for a half-plane with two kinds of boundary conditions were first obtained by Galin /1/. The problem was formulated for a composite plane with three kinds of boundary conditions, whose solution is obtained in quadratures in the case of one slipping section /4/. However, as shown in /3, 5/, the method described in /4/ does not result in an exact solution for a large number of sections.

1. The Hilbert-Riemann problem for a plane with slots. We consider a combination Hilbert-Riemann boundary value problem for a piecewise-analytic function $\Phi(z)$ in the complex $z = x + iy$ plane with boundary lines LUM /2/:

$$\operatorname{Im}[p^{\pm}(x)\Phi^{\pm}(x)] = f^{\pm}(x), \quad p^{\pm}(x) \neq 0, \quad x \in L = L^1 \cup L^2 \tag{1.1}$$

$$\Phi^+(x) = G(x)\Phi^-(x) + g(x), \quad x \in M = M^1 \cup M^2, \tag{1.2}$$

$$L \cap M = 0, \quad L \cup M = (-\infty, \infty)$$

in the special case which is important for applications when $G(x) = G = \text{const}$, $x \in M^1$, $G(x) = 1$, $x \in M^2$ the function $p^{\pm}(x) = p(x)$ takes real values on L^1 and pure imaginary values on L^2. Let L consist of α' half-open, α'' open and $R - \alpha' - \alpha''$ closed intervals $\langle a_k, b_k\rangle$, $k = 1, 2, \ldots, R$, M^1 from the segments $[s_k, t_k]$, $k = 1, 2, \ldots, Q$ of the real axis $a_1 < b_1 < \ldots < b_R$, $s_1 < t_1 < \ldots < t_Q$. Without loss of generality it can obviously be assumed that $p(x) \equiv 1$ on L^1 and $p(x) \equiv i$ on L^2. We will assume that every boundary point of the outline L does not belong to L except in the case when it is a boundary point of M^1. Let the intervals $\langle a_k, b_k\rangle$ contain N_k inner nodes $x = d_{kl}$ that are simultaneously boundaries for L^1 and L^2 at which the function $p(x)$ undergoes a discontinuity $d_{kl} < d_{k,l+1}$, the total number of inner nodes equals N on L and the functions $f^{\pm}(x)$ and $g(x)$ satisfy the Hölder condition.

We will seek the solution of problem (1.1) and (1.2) in the broadest class h_0 /6/ of piecewise-analytic functions tending to zero at infinity by using the canonical solution $X(z)$ of the corresponding homogeneous problem by setting /2/

$$X(z) = Z(z)\, e^{i\psi(z)} \prod_{j=1}^{R} (z - b_j)^{-\alpha_j} \prod_{j=1}^{R-1} (z - c_j)^{-\beta_j} \tag{1.3}$$

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$$Z(z) = \prod_{k=1}^{Q} (z - s_k)^{-1/2+i\gamma} (z - t_k)^{-1/2-i\gamma}$$

$$\psi(z) = \frac{1}{2\pi i} \int_L \left\{ \frac{Y(z)\,[h^+(t) + h^-(t)]}{Y^+(t)} + h^+(t) - h^-(t) \right\} \frac{dt}{t - z}$$

$$\gamma = \frac{\ln(-G)}{2\pi}$$

$$Y(z) = \prod_{k=1}^{R} (z - a_k)^{1/2} (z - b_k)^{1/2}, \quad Y(z) = z^R + O(z^{R-1}), \quad z \to \infty$$

$$h^\pm(t) = \pi n_k^\pm - \arg Z^\pm(t) + \sum_{j=1}^{R-1} \beta_j \arg (t - c_j)^\pm +$$
$$\sum_{j=1}^{R} \alpha_j \arg (t - b_j)^\pm + \pi m^\pm(t) + {}^1/_2 (1 - s)\pi,$$
$$t \in \langle a_k, b_k \rangle \cap L^s, \quad k = 1, \ldots, R; \quad s = 1, 2$$

Here $n_k^\pm$, α_k, $\beta_k \neq 0$ are integers, c_k are complex numbers, $m^\pm(t)$ are integer functions to be determined that can have jumps on the edges of the slots for $x = d_{kl} \pm i0$, $m^\pm(a_k) = 0$; $Z(z)$ is the canonical solution of the homogeneous Riemann problem (1.2) in h_0; $\psi(z)$ is the solution of the Dirichlet problem $\operatorname{Re}\psi^\pm(x) = h^\pm(x)$, $x \in L$ bounded at all the nodes as well as at infinity because of additional conditions imposed on the function $h^\pm(t)$

$$\int_L \frac{h^+(t) + h^-(t)}{Y^+(t)} t^{j-1}\, dt = 0, \quad j = 1, \ldots, R-1 \tag{1.4}$$

Unlike the Riemann problem /6/, it is impossible to construct the canonical solution in the Hilbert-Riemann problem in the general case in the same class h_m as the general solution.

The asymptotic form of the function $X(z)$ at nodes of the line L can be written in the form $X(z) = O[(z - d)^\zeta]$, $z \to d$, where $\zeta = \lambda_k$ for $d = a_k$, $\zeta = \nu_k$ for $d = b_k$, $\zeta = \gamma_{kl}^\pm$ for $d = d_{kl} \pm i0$ and the following equalities hold

$$\lambda_k = \delta_k + \omega_k - {}^1/_2 w_k^-, \quad \nu_k = \Delta_k + \varepsilon_k - \omega_k + {}^1/_2 w_k^- - \alpha_k, \tag{1.5}$$
$$w_k^- = n_k^+ - n_k^-$$
$$\Delta_k = {}^1/_2 [m^+(b_k) - m^-(b_k)], \quad \gamma_{kl}^\pm = \pm [\theta(d_{kl}) - m^\pm(d_{kl} + 0) +$$
$$m^\pm(d_{kl} - 0)]$$
$$\omega_k = \theta_k - \frac{1}{2\pi} \arg \frac{Z^+(x)}{Z^-(x)}, \quad x \in \langle a_k, b_k \rangle, \quad \theta_k = \sum_{j=1}^{k-1} \alpha_j \; (k > 1),$$
$$\theta_1 = 0$$
$$\theta(t) = \pi^{-1} \arg \{p(t + 0)\,[p(t - 0)]^{-1}\}$$

Here $\delta_k = -{}^1/_2$ $(\delta_k = 0)$ if $a_k \in M^1$ $(a_k \overline{\in} M^1)$ and $\varepsilon_k = -{}^1/_2$ $(\varepsilon_k = 0)$ if $b_k \in M^1$ $(b_k \overline{\in} M^1)$. Since α_k are integers then θ_k and ω_k are also integers and the quantities $\theta(t)$, λ_k, ν_k, $\gamma_{kl}^\pm$ are multiples of ${}^1/_2$.

Setting $\gamma_{kl}^\pm \equiv -{}^1/_2$ and taking into account that $m^\pm(a_k) = 0$, $m^\pm(d_{kl} + 0) = m^\pm(d_{k,l+1} - 0)$, for sequential calculations in l of the functions $m^\pm(t)$ we obtain a recursion formula for any k $m^\pm(d_{kl} + 0) = m^\pm(d_{kl} - 0) + E\{\theta(d_{kl})\} + {}^1/_2 \pm {}^1/_2$ where $E\{t\}$ is the integer part of t. Hence and from (1.5) it follows that

$$\Delta_k = {}^1/_2 N_k, \quad \alpha_k = \varepsilon_k + \delta_k - \lambda_k - \nu_k + {}^1/_2 N_k \tag{1.6}$$

Let r_1 be the index of the degree of growth of the function

$$\prod_{k=1}^{R} (z - b_k)^{\alpha_k}$$

as $z \to \infty$, r_2 the number of nodes at which the function $X(z)$ has no singularities, and let us introduce a new notation of the intervals $\langle a_k, b_k \rangle$. Let L_{ns}^t be half-open $(t = 1)$, open $(t = 2)$ or closed $(t = 3)$ intervals $\langle a_n, b_n \rangle$ with an odd $(s = 1)$ or even $(s = 2)$ number of inner nodes equal to N_{ns}^t, n varies between 1 and α_s^t, and L^* **is the union of all intervals** $L_{n1}^1, L_{n2}^2, L_{n2}^3$. Then a unique triplet n, s, t can be set in correspondence to each k, and the corresponding number α_k can be denoted by α_{ns}^t. We set $\lambda_k = \nu_k = -{}^1/_2$ for $\langle a_k, b_k \rangle \in L^*$, $\lambda_k = -{}^1/_2$, $\nu_k = 0$ for $\langle a_k, b_k \rangle \in L \setminus L^*$. Using (1.6) and the equalities

$$\sum_{s=1}^{2}\sum_{t=1}^{3}\alpha_s{}^t=R,\quad \sum_{s=1}^{2}\sum_{t=1}^{3}\sum_{n=1}^{\alpha_s^t}N_{ns}^t=N,\ \alpha_1{}^1+\alpha_2{}^1=\alpha',\ \alpha_1{}^2+\alpha_2{}^2=\alpha''$$

we obtain all the numbers α_k and r_q in the form

$$\begin{aligned}&\alpha_{n1}^1={}^1/_2(N_{n1}^1+1),\quad \alpha_{n1}^2={}^1/_2(N_{n1}^2-1),\quad \alpha_{n1}^3={}^1/_2(N_{n1}^3+1)\\ &\alpha_{n2}^1={}^1/_2N_{n2}^1,\quad \alpha_{n2}^2={}^1/_2N_{n2}^2,\quad \alpha_{n3}^3={}^1/_2N_{n2}^3+1,\\ &r_2=\alpha_1{}^2+\alpha_1{}^3+\alpha_2{}^1\\ &r_1=\sum_{k=1}^{R}\alpha_k=\sum_{s=1}^{2}\sum_{t=1}^{3}\sum_{n=1}^{\alpha_s^t}\alpha_{ns}^t={}^1/_2N+R-\alpha''-\alpha_2{}^1-\\ &\quad {}^1/_2(\alpha_1{}^1+\alpha_1{}^2+\alpha_1{}^3)\end{aligned}\tag{1.7}$$

Further operations exactly duplicate the procedure for solving the Dirichlet-Riemann problem /7/. The quantities θ_k, ω_k, $w_k^-=2(\delta_k+\omega_k-\lambda_k)$ are calculated sequentially by means of (1.5). The integers $w_k^+=n_k^++n_k^-$ of given evenness and coinciding with w_k^- and the complex numbers c_k. The affixes of points arranged on the curves S_k whose ends are the points a_k, b_k, $k=1,\ldots,R-1$; $w_R^+=w_R^-$, are found from the system of transcendental Eqs. (1.4). Knowing $w_k\pm$ the integers $n_k\pm$ can be found but they do not occur in the solution (1.3) separately. Rational methods for selecting β_k and S_k are examined in /7/.

To be specific let $\beta_k\equiv 1$, the function $X(z)$ has only simple poles at the points $z=c_k$, and S_k is a semicircle in the half-plane $\operatorname{Im} z>0$. Then by the construction of (1.3) the asymptotic form $X(z)$ has the following form at infinity

$$X(z)=O(z^{-r}),\quad r=Q+R+r_1-1\tag{1.8}$$

The general solution of problem (1.1) and (1.2) in the class h_0 is expressed by the formulas /7/

$$\Phi(z)=X(z)[\Phi_1(z)+\Phi_2(z)]\tag{1.9}$$

$$\begin{aligned}&\Phi_1(z)=\frac{1}{2\pi i}\int\limits_M\frac{g(t)\,dt}{X^+(t)(t-z)},\quad \Phi_2(z)=\frac{Y_0(z)}{2\pi}\int\limits_L\frac{f_2^+(t)+f_2^-(t)}{Y_0^+(t)(t-z)}dt+\\ &\quad\frac{1}{2\pi}\int\limits_L\frac{f_2^+(t)-f_2^-(t)}{t-z}dt+P_{r-1}(z)+iQ_s(z)Y_0(z),\\ &s=r+r_2-R-1\\ &f_2\pm(x)=f\pm(x)[X\pm(x)]^{-1}-\operatorname{Im}\Phi_1(x),\quad x\in L;\\ &Y_0(z)=Y(z)\prod_{n=1}^{r_2}(z-b_n{}^*)^{-1}\end{aligned}$$

Here $P_r(z)$ and $Q_s(z)$ are polynomials of degree r and s with real coefficients, $b_n{}^*$, $n=1,\ldots,r_2$ are the right ends of those intervals $\langle a_n, b_n\rangle$ at which the function $X(z)$ is bounded. By virtue of (1.9), (1.8), and (1.7) the total number of coefficients in both polynomials equals $r+s+1$ or $2Q+3R+N-\alpha'-2\alpha''-2$. Here $2R-2$ coefficients are removed when eliminating the poles of the function $\Phi(z)$ at the points $z=c_k$ when solving the system of equations $\Phi_1(c_k)+\Phi_2(c_k)=0$, $k=1,\ldots,R-1$. Therefore, the number of arbitrary real constants in the solution obtained equals $2Q+R+N-\alpha'-2\alpha''$ and is always positive since $2Q\geqslant\alpha'+2\alpha''$, $r\geqslant 1$, $s\geqslant 0$. An analogous calculation involving the orthogonality conditions of the free terms can be made for solutions in any class h_m /6/.

2. The Hilbert problem for a plane with slots. If there is no second condition in problem (1.1) and (1.2), its solution (1.3)-(1.9) is simplified: $Z(z)\equiv 1$, $Q=0$, $\alpha'=\alpha''=0$ the number of arbitrary constants becomes equal to $N+R$ and the points c_k determined by the system of transcendental Eqs.(1.4) agree with the nodes a_k. Let us construct a still simpler solution in which conditions (1.4) do not occur.

In the Hilbert problem (1.1) written in the form

$$\operatorname{Im}[p_s\Phi\pm(x)]=f_s\pm(x),\quad s=1,2;\ p_1=1,\quad p_2=i,\quad x\in L^s\tag{2.1}$$

let the lines L^1 and L^2 respectively, consist of l^1 and l^2 segments $[a_k{}^1, b_k{}^1]$ and $[a_k{}^2, b_k{}^2]$ and let them have N common nodes as in Sect.1; obviously $l^1+l^2=N+R$. The solution of the Dirichlet problem for a plane with slots $\operatorname{Im}X_1\pm(x)=0$, $x\in L^1$ having the form

$$X_s(z)=i\Big[\prod_{k=1}^{l^s}(z-a_k{}^s)(z-b_k{}^s)\Big]^{-1/2},\quad X_s(z)\sim z^{-l_s},\quad z\to\infty\tag{2.2}$$

for $s = 1$ can be taken as the canonical solution $X(z)$ of the homogeneous problem (2.1) for $f_s^{\pm}(x) \equiv 0$. Indeed, the function $X(z) = X_1(z)$ is pure imaginary on the Ox axis beyond L^1; consequently, as condition (2.1) demands, $\operatorname{Re} X(x) = 0$ on L^2. Now by virtue of (2.1) the function $F(z) = \Phi(z)[X_1(z)]^{-1}$ can be found by solving the Dirichlet problem for a plane with slots

$$\operatorname{Im} F^{\pm}(x) = f_s^{\pm}(x)[p_s X_1^{\pm}(x)]^{-1}, \quad x \in L^s, \quad s = 1, 2 \tag{2.3}$$

The function $\Phi(z)$ in this class h_0 should have a power-law singularity with exponent $-{}^1/_2$ at all nodes a_k^s, b_k^s and should decrease as z^{-1} at infinity. Hence, and from (2.2) it follows that the solution of problem (2.3) must be sought in the class of functions growing as z^{η}, $\eta = l^1 - 1$ as $z \to \infty$, bounded at the ends $a_1^*, a_2^*, \ldots, a_{\xi}$ of R of the slots $[a_k, b_k]$ in which ξ_1 of some points a_k and b_k coincide, respectively, with a_n^1 and b_n^1 and having an integrable infinity at the remaining $2R - \xi$ ends a_k and b_k. We write this solution down by using the function (2.2) for $s = 1$ and 2 and therefore eliminating all quantities $z - a_k^*$

$$F(z) = \sum_{s=1}^{2} \frac{1}{2\pi p_s} \int_{L^s} \left\{ \frac{X_2(z)\, X_1^+(t)}{X_1(z)\, X_2^+(t)} \left[\frac{f_s^+(t)}{X_1^+(t)} + \frac{f_s^-(t)}{X_1^-(t)} \right] + \frac{f_s^+(t)}{X_1^+(t)} - \frac{f_s^-(t)}{X_1^-(t)} \right\} \frac{dt}{t - z} + P_{\eta}(z) + iQ_{\theta}(z) X_2(z)[X_1(z)]^{-1}$$

$$\theta = R + \eta - \xi$$

Taking account of the equalities

$$\Phi(z) = X_1(z) F(z), \quad X_s^-(t) = (-1)^{s+q+1} X_s^+(t), \quad t \in L^q, \quad R = l^1 + l^2 - N, \quad \xi = 2l^1 - N$$

we hence obtain the general solution of problem (2.1)

$$\Phi(z) = \sum_{s=1}^{2} \frac{1}{2\pi p_s} \sum_{q=1}^{2} \int_{L^q} \frac{X_q(z)}{X_q^+(t)} [f_s^+(t) + (-1)^{s+q} f_s^-(t)] \frac{dt}{t - z} + P_{\eta}(z) X_1(z) + iQ_{\theta}(z) X_2(z), \quad \eta = l^1 - 1, \quad \theta = l^2 - 1 \tag{2.4}$$

We note that the number of arbitrary constants $l^1 + l^2$ or $N + R$ and the form of this solution for fixed l^1 and l^2 is independent of the number of common nodes N of the lines L^1 and L^2; for instance, merger of any slots from L^1 and L^2 is not reflected in (2.4).

If $f_s^+(x) = f_s^-(x)$, $s = 1, 2$, the solution (2.4) of the Hilbert problem (2.1) separates into the sum of solutions of two separate Dirichlet problems (2.1) for $s = 1$ and $s = 2$:

$$\Phi(z) = X_1(z) \left[\frac{1}{\pi} \int_{L^1} \frac{f_1^+(t)\, dt}{X_1^+(t)(t - z)} + P_{\eta}(z) \right] + iX_2(z) \left[\frac{1}{\pi} \int_{L^2} \frac{f_2^+(t)\, dt}{X_2^+(t)(t - z)} + Q_{\theta}(z) \right] \tag{2.5}$$

3. The contact problem for an elastic half-plane.

Let $L_{km} = \langle a_{km}, b_{km} \rangle$, $k = 1, \ldots, k_m$, $m = 1, 2$ be any open, half-open, or closed intervals, $L_{k3} = [a_{k3}, b_{k3}]$, $k = 1, 2, \ldots, k_3$ are segments of the Ox axis of an xOy Cartesian system of coordinates moving at a constant subsonic velocity c in the direction of the Ox axis relative to the elastic half-plane $-\infty < x < \infty$, $y < 0$ and $a_{km} < b_{km} < a_{k+1,m}$ for all k and m.

We write down the boundary conditions

$$u' = u_0(x), \quad x \in L_2 \cup L_3; \quad v' = v_0(x), \quad x \in L_1 \cup L_3; \quad L_m = \bigcup_{k=1}^{k_m} L_{km}$$
$$\tau_{xy} = \tau(x), \quad x \in L_1 \cup L_4; \quad \sigma_y = \sigma(x), \quad x \in L_2 \cup L_4; \quad L_k \cap L_l = 0, \quad k \neq l \tag{3.1}$$

corresponding to sliding contact of the stamp on L_1, to adhesion of the flexible inextensible cover-plane on L_2, to total adhesion of the stamp and half-plane on L_3, to the assignment of the stresses on L_4, the complements $L_1 \cup L_2 \cup L_3$ to the Ox axis. We will consider that the given functions satisfy the Hölder condition, and any boundary point of L_m, $m = 1, 2$, does not belong to L_m only in case it belongs to L_3. We set the rotation and compression equal to zero at infinity, we give the jumps χ_{km}, $m = 1, 2$ on all segments $[a_{k1}, b_{k1}]$ and $[a_{k2}, b_{k2}]$ in the open intervals (a_{km}, b_{km}) and the quantities Y_{k1} and X_{k2}, respectively, in $[a_{k3}, b_{k3}]$ for those k for which a_{k3} is not a boundary point of some interval $(a_{n,m}, b_{nm})$, $m = 1, 2$ the quantities X_{k3}, Y_{k3}, where X_{km}, Y_{km} are the principal shear and normal stress vectors on L_{km}, $\chi_{k1} = u(b_{k1}) - u(a_{k1})$, $\chi_{k2} = v(b_{k2}) - v(a_{k2})$. The total number of these arbitrary force and kinematic parameters of the problem obviously equals $k_1 + k_2 + 2k_3 - \alpha' - 2\alpha''$, where α' is the number of half-open, and α'' is the number of open intervals in $L_1 \cup L_2$.

We will seek the solution of the problem (3.1) in the Galin form /1, 8/

$$\mu u' = -\mathrm{Re}\,[\varphi_1(z_1) + q_2\varphi_2(z_2)], \quad \mu v' = \mathrm{Im}\,[q_1\varphi_1(z_1) + \varphi_2(z_2)]$$
$$\sigma_y = 2\mathrm{Re}\,[q\varphi_1(z_1) + q_2\varphi_2(z_2)], \quad \tau_{xy} = 2\mathrm{Im}\,[q_1\varphi_1(z_1) + q\varphi_2(z_2)], \quad z_s = x + iq_s y$$
$$q_s = \sqrt{1 - c^2 c_{s*}^{-2}}, \quad 2q = 1 + q_2^2, \quad c_{1*}^2 = 2(1-\nu)(1-2\nu)^{-1} c_{2*}^2, \quad c_{2*}^2 = \mu\rho^{-1} \tag{3.2}$$

where μ is the shear modulus, ν is Poisson's ratio, ρ is the material density, c_{1*} and c_{2*} are the longitudinal and transverse wave propagation velocities, $\varphi_s(z)$, $s = 1, 2$ are functions analytic in the half-plane $\mathrm{Im}\, z < 0$, $z = x + iy$, tending to zero in it as $z \to \infty$, and $' = \partial/\partial x$.

We will introduce a representation of these functions in terms of one function $\Phi(z)$ that is piecewise-analytic in the z plane with boundary line $y = 0$. Requiring that the function $\Phi(z)$ satisfy the Hilbert condition in $L_1 \cup L_2$ and the Riemann condition in $L_3 \cup L_4$ as in /7/, we obtain

$$\varphi_s(z) = q_s^{-1/2}[(-1)^{s+1} R^+\Phi(z) + R^-\bar{\Phi}(z)], \quad R^\pm = \sqrt{q_1 q_2} \pm q, \quad s = 1, 2 \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we arrive at the combined Hilbert-Riemann problem (1.1) and (1.2) in which

$$L^s = L_s, \quad s = 1, 2$$
$$M^1 = L_3, \quad M^2 = L_4, \quad G = -G^+/G^-, \quad G^\pm = R^\pm (Q^\pm)^{-1},$$
$$Q^\pm = 1 \pm \sqrt{q_1 q_2}$$
$$f^\pm(x) = W_2^{-1} [\mp {}^1/_2 (G^\mp)^{-1} \tau(x) - \mu v_0(x)], \quad x \in L^1, \quad W_s = 2q_s^{1/2}(1 - q)$$
$$f^\pm(x) = W_1^{-1} [\pm {}^1/_2 (G^\mp)^{-1} \sigma(x) - \mu u_0(x)], \quad x \in L^2$$
$$g(x) = -\mu (R^- Q^+)^{-1} [q_1^{1/2} u_0(x) + iq_2^{1/2} v_0(x)], \quad x \in M^1$$
$$g(x) = {}^1/_2 R_*^{-1} [q_1^{1/2}\sigma(x) - iq_2^{1/2}\tau(x)], \quad x \in M^2$$
$$R_* = R^+ R^- = q_1 q_2 - q^2$$
$$Q_* = Q^+ Q^- = 1 - q_1 q_2,$$

R_* and Q_* are Rayleigh functions for the free and clamped half-plane and the number of arbitrary constants equals the number of parameters of the contact problem since $k_1 + k_2 = N + R$, $k_3 = Q$.

It is convenient to use the solution in the form (3.3) in the case when a section of the boundary L_4 contains one or two semi-infinite intervals. If the section L_3 is extended to infinity then by using the representation

$$\varphi_s(z) = q_s^{-1/2} [(-1)^s Q^+\Phi(z) + Q^-\bar{\Phi}(z)], \quad s = 1, 2 \tag{3.4}$$

it is convenient to reduce the problem (3.1) to the boundary value problem (1.1) and (1.2) by replacing L_3 by L_4, L_4 by L_3 and the functions $f^\pm(x), G(x)$, and $g(x)$ by $\pm G^\mp f^\pm(x), G^{-1}G(x)$, and $G^- g(x)$.

It is possible to pass to the limit in (3.2) and (3.3) as $c \to 0$ to solve the static problem. However, in this case it is simpler to construct the solution on the basis of **Muskhelishvili potentials** by following /7/.

4. The motion of a flexible cover-plane and stamp on a half-plane.

On the boundary $y = 0$ of an elastic half-plane $y < 0$ let a rigid stamp $x \in L_1 = [a_1, b_1]$ and a flexible inextensible coverplate $x \in L_2 = [a_2, b_2]$, $a_1 < b_1 < a_2 < b_2$ move together with a xOy coordinate system at a constant subsonic velocity c. A normal compressive force P is applied to the stamp, there is no contact friction, and a distributed normal load $\sigma_y = \sigma(x)$ and a longitudinal force T are applied to the cover-plate attached to the half-plane (a caterpillar track).

The boundary conditions of this problem

$$v' = \tau_{xy} = 0, \quad x \in L_1; \quad u' = 0, \quad \sigma_y = \sigma(x), \quad x \in L_2; \quad \tau_{xy} = \sigma_y = 0, \quad x \in L_4 \tag{4.1}$$

are identical with conditions (3.1) where there is no section L_3. Following (3.2)-(3.5) the Hilbert problem for a plane with the slots L_1 and L_2 can be written as follows:

$$\operatorname{Im}\Phi^{\pm}(x) = 0; \quad x \in L_1; \quad \operatorname{Re}\Phi^{\pm}(x) = \pm [G^{\pm}W_2]^{-1}\sigma(x), \quad x \in L_2 \tag{4.2}$$

According to (2.2) and (2.4), its solution has the form

$$\Phi(z) = \Phi_0(z) + iC_1Y_1^{-1}(z) + C_2Y_2^{-1}(z) \tag{4.3}$$

$$Y_s(z) = \sqrt{(z-a_s)(z-b_s)}, \quad s = 1, 2$$

$$2\pi R\Phi_0(z) = \frac{\sqrt{q_1}}{Y_1(z)}\int_{L_2}\frac{Y_1(t)\sigma(t)\,dt}{t-z} - \frac{q_1q_2-q}{W_2}\int_{L_2}\frac{Y_2^+(t)\sigma(t)\,dt}{t-z} \tag{4.4}$$

Determining the arbitrary constants in terms of the given forces P and T, we obtain

$$C_1 = \frac{(Y-P)\sqrt{q_2}}{4\pi R}, \quad C_2 = \frac{T\sqrt{q_1}}{4\pi R}, \quad Y = \frac{1}{2\pi}\int_{L_2}\sigma_y\,dx$$

If $\sigma(t) = \sigma_0 = \text{const}$, the solution is expressed in terms of elementary functions. Evaluating the tabulated integrals, we obtain from (4.4)

$$\frac{2\pi R}{\sigma_0}\Phi_0(z) = \frac{\sqrt{q_1}}{Y_1(z)}\Bigg[Y_1(b_2) - Y_1(a_2) + (2z-a_1-b_1)\times$$
$$\ln\frac{\sqrt{b_2-a_1}+\sqrt{b_2-b_1}}{\sqrt{a_2-a_1}+\sqrt{a_2-b_1}}\Bigg] + 2\sqrt{q_1}\ln\times$$
$$\left[\frac{\sqrt{(a_2-b_1)(z-a_1)}+\sqrt{(a_2-a_1)(z-b_1)}}{\sqrt{(b_2-b_1)(z-a_1)}+\sqrt{(b_2-a_1)(z-b_1)}}\sqrt{\frac{z-b_2}{z-a_2}}\right] -$$
$$\frac{\pi(q_1q_2-q)}{W_2}\left[1-\frac{2z-a_2-b_2}{2Y_2(z)}\right]$$

If $\sigma_0 = 0$ then $\Phi_0(z) \equiv 0, Y = 0$ and by virtue of (2.5) the solution will separate into the sum of solutions of two Dirichlet problems. This is in agreement with the fact that $u(x,0) = H(0)$ and $v(x,0) = H(0)$, respectively, in the solutions of the Flamant and Cerruti problems for forces applied at the point $z = 0$ (in both statics and stationary dynamics) where $H(x)$ is the Heaviside function. The contact stresses under the stamp and cover-plate in this case also do not, naturally, experience any interactive influence

$$\sigma_y = -\pi^{-1}P(x-a_1)^{-1/2}(b_1-x)^{-1/2}, \quad x \in L_1$$
$$\tau_{xy} = \pi^{-1}T(x-a_2)^{-1/2}(b_2-x)^{-1/2}, \quad x \in L_2$$

5. The motion of slots in a composite elastic plane. Let $L_{km} = \langle a_{km}, b_{km}\rangle$, $k = 1, 2, \ldots, k_m$, $m = 1, \ldots, 5$ be intervals of the Ox axis of a Cartesian system xOy moving at a Sub-Rayleigh constant velocity c relative to a composite elastic plane, $L_m = U_{k=1}^{k_m}L_{km}$, L_6 is the complement $L_1\cup, \ldots, \cup L_5$ to the Ox axis. The line separating the elastic materials of the plane is superposed on the Ox axis, and magnitudes referred to the half-planes $y > 0$ and $y < 0$ are denoted by the subscripts $j = 1$ and $j = 2$. Let the half-planes be completely adherent in the closed intervals $L_1 = [a_{k1}, b_{k1}]$; there are slots on the other sections between them, where the slots are open on L_2 and wedging stamps are imbedded in them, there is no friction; the slot edges in L_3 adhere to the flexible inextensible cover-planes; sliding conditions are posed on L_4, "anti-sliding" contact of the edges on L_5; and stresses are applied to the slot edges on L_6. We will consider that the boundary point of any interval L_{km}, $m = 2, \ldots, 5$ does not belong to L_{km} only when it belongs to L_1. The total number of half-open intervals in $L_2, \ldots, L_5$ is denoted by α' and the open intervals by α''. We write down the boundary conditions of the problem

$$[\sigma_y(x)] = \sigma^\circ(x), \quad [\tau_{xy}(x)] = \tau^\circ(x), \quad [u'(x)] = u^\circ(x), \tag{5.1}$$
$$[v'(x)] = v^\circ(x), \quad x \in L_1$$
$$\tau_{xyj} = \tau_j^\circ(x), \quad v_j'(x) = v_j^\circ(x), \quad x \in L_2$$
$$\sigma_{yj}(x) = \sigma_j^\circ(x), \quad u_j'(x) = u_j^\circ(x), \quad x \in L_3$$
$$\tau_{xyj} = \tau_j^\circ(x), \quad [v'(x)] = v^\circ(x), \quad [\sigma_y(x)] = \sigma^\circ(x), \quad x \in L_4$$
$$\sigma_{yj}(x) = \sigma_j^\circ(x), \quad [u'(x)] = u^\circ(x), \quad [\tau_{xy}(x)] = \tau^\circ(x), \quad x \in L_5$$
$$\sigma_{yj}(x) = \sigma_j^\circ(x), \quad \tau_{xyj}(x) = \tau_j^\circ(x), \quad x \in L_6; \quad [f(x)] = f_1(x) - f_2(x)$$

Taking account of the kinematic contact conditions, we additionally give the principal vector X^∞, Y^∞ of the stress field at infinity for $y > 0$; we give the displacement jumps $[u(a_{k1})] = u_{k1}^\circ$ and $[v(a_{k1})] = v_{k1}^\circ$ at points a_{k1} that are not boundary point for any intervals (a_{lm}, b_{lm}), $m = 2, \ldots, 5$; we give the normal force Y_k applied to the stamp on each segment $[a_{k2}, b_{k2}]$ and the jump $[v(a_{k2})] = v_{k2}^\circ$, two longitudinal forces X_{kj} applied to the coverplates are on $[a_{k3}, b_{k3}]$; the jump $[v(a_{k4})] = v_{k4}^\circ$ is on $[a_{k4}, b_{k4}]$, and the jump $[u(a_{k5})] = u_{k5}^\circ$ is on $[a_{k5}, b_{k5}]$. The total number of these additional quantities equals $2(k_1 + k_2 + k_3) + k_4 + k_5 - \alpha' - 2\alpha''$.

We will seek the solution in each half-plane in the form (3.2)

$$\begin{gathered}\mu_j u_j' = -\mathrm{Re}\,[\varphi_{j1}(z_{j1}) + q_{j2}\varphi_{j2}(z_{j2})], \quad \mu_j v_j' = \\ \mathrm{Im}\,[q_{j1}\varphi_{j1}(z_{j1}) + \varphi_{j2}(z_{j2})] \\ \sigma_{yj} = 2\mathrm{Re}\,[q_j\varphi_{j1}(z_{j1}) + q_{j2}\varphi_{j2}(z_{j2})], \quad \tau_{xyj} = 2\mathrm{Im}\,[q_{j1}\varphi_{j1}(z_{j1}) + \\ q_j\varphi_{j2}(z_{j2})] \\ q_{jk} = \sqrt{1 - c^2 c_{jk}^{-2}}, \quad 2q_j = 1 + q_{j2}^2, \quad z_{jk} = x + iq_{jk}y, \quad k, j = 1, 2\end{gathered} \tag{5.2}$$

where c_{j1} and c_{j2} are the longitudinal and transverse wave propagation velocities in the j-th elastic medium.

We select the functions $\varphi_{jk}(z)$ by again being guided by the Hilbert and Riemann conditions (μ_j is the shear modulus)

$$\begin{gathered}\varphi_{jk}(z) = \sum_{s=1}^{2} (-1)^{s(1+j)} [q_{jk}^{s-} F_s(z) - (-1)^k q_{jk}^{s+} \bar{F}_s(z)] \\ q_{j1}^{s\pm} = \frac{q_{j2}}{p_{j2}^{\,s}} \pm \frac{q_j}{p_{j1}^{\,s}}, \quad q_{j2}^{s\pm} = \frac{q_{j1}}{p_{j1}^{\,s}} \pm \frac{q_j}{p_{j2}^{\,s}}, \quad p_{jk}^1 = \sqrt{p_k}, \\ R_j = q_{j1}q_{j2} - q_j^2 \\ p_{jk}^2 \equiv p_{jk} = \varkappa^{j-1} q_{jk}(1 - q_j) R_j^{-1}, \quad \varkappa = \mu_1\mu_2^{-1}, \quad p_k = p_{1k} + p_{2k}\end{gathered} \tag{5.3}$$

Substituting (5.2) into (5.1), we obtain two unrelated combined boundary value problems for the piecewise-analytic functions $F_1(z)$ and $F_2(z)$.

For the first function this is the Hilbert-Riemann problem (1.1) and (1.2) for

$$\begin{gathered}\Phi(z) = F_1(z), \ L^1 = L_2 \cup L_4, \ L^2 = L_3 \cup L_5, \ M^1 = L_1, \ M^2 = L_6 \\ f^\pm(x) = {}^1\!/_2 \mu_1 p_1^{-1/2} v^0(x) - p_1^{-1/2}[h_1\tau_1^0(x) - h_2\tau_2^0(x)] \pm \\ p_2^{-1/2}[p_{12}\tau_1^0(x) + p_{22}\tau_2^0(x)], \quad x \in L^1 \\ f^\pm(x) = -{}^1\!/_2 \mu_1 p_2^{-1/2} u^\circ(x) - p_2^{-1/2}[h_1\sigma_1^\circ(x) - h_2\sigma_2^\circ(x)] \pm \\ p_1^{-1/2}[p_{11}\sigma_1^\circ(x) + p_{21}\sigma_2^\circ(x)], \quad x \in L^2 \\ G = G^-(G^+)^{-1}, \quad G^\pm = h_1 - h_2 \pm \sqrt{p_1p_2}, \\ h_j = \varkappa^{j-} R_j^{-1}(q_{j1}q_{j2} - q_j), \quad j = 1, 2 \\ g(x) = -(G^+)^{-1}\{p_1^{1/2}[\mu_1 u^\circ(x) + 2(h_1p_{21} + h_2p_{11})\sigma^\circ(x) - \\ ip_2^{1/2}[\mu_1 v^\circ(x) + 2(h_1p_{22} + h_2p_{12})\tau^\circ(x)\}, \quad x \in M^1 \\ g(x) = 2p_1^{-1/2}[p_{11}\sigma_1^\circ(x) + p_{21}\sigma_2^\circ(x)] + 2ip_2^{-1/2}[p_{12}\tau_1^\circ(x) + p_{22}\tau_2^\circ(x)], \\ x \in M^2\end{gathered}$$

Its solution (1.9), (1.3) and (1.4) has $2k_1 + k_2 + k_3 + k_4 + k_5 - \alpha' - 2\alpha''$ arbitrary constants.

The boundary value problem for the function $F_2(x)$ has the form

$$\begin{gathered}\mathrm{Im}\,[p_0(x)F_2^\pm(x)] = f_0^\pm(x), \quad x \in L_2 \cup L_3, \quad p_0(x) = -i^n, \\ x \in L_n \\ F_2^+(x) - F_2^-(x) = g_2(x), \quad x \in M_0 = L_1 \cup L_4 \cup L_5 \cup L_6 \\ f_0^\pm(x) = {}^1\!/_2 \mu_1 p_1^{-1}[p_{21}v_1^\circ(x) + p_{11}v_2^\circ(x)] - p_1^{-1}[h_1p_{21}\tau_1^\circ(x) + \\ h_2p_{11}\tau_2^\circ(x)] \pm p_{12}p_{22}p_2^{-1}\tau^\circ(x), \quad x \in L_2; \\ f_0^\pm(x) = -\mu_1p_2^{-1}[p_{22}u_1^\circ(x) + p_{12}u_2^\circ(x)] - \\ p_2^{-1}[h_1p_{22}\sigma_1^\circ(x) + h_2p_{12}\sigma_2^\circ(x)] \pm p_1^{-1}p_{11}p_{21}\sigma^\circ(x), \quad x \in L_3 \\ g_2(x) = 2p_1^{-1}p_{11}p_{21}\sigma^\circ(x) + 2ip_2^{-1}p_{12}p_{22}\tau^\circ(x) \\ \sigma^\circ(x) = \sigma_1^\circ(x) - \sigma_2^\circ(x), \quad \tau^\circ(x) = \tau_1^\circ(x) - \tau_2^\circ(x)\end{gathered} \tag{5.4}$$

We set $F_2(z) = \Phi_2(z) + \Phi_*(z)$ where $\Phi_*(z)$ is the solution of the problem of a jump $\Phi_*^+(x) - \Phi_*^-(x) = g_2(x)$, $x \in M_0$, having the form

$$\Phi_*(z)=\frac{1}{2\pi i}\int_{M_*}\frac{g_2(t)\,dt}{t-z}$$

Then we obtain the Hilbert problem (2.1) for the function $\Phi(z)=\Phi_2(z)$ in which $L^1=L_2$, $L^2=L_3$, $f_s^{\pm}(x)=f_0^{\pm}(x)-\mathrm{Im}\,[p_0(x)\,\Phi_*(x)]$, $x\in L^s$, $s=1,2$. Its solution (2.4) contains k_2+k_3 arbitrary constants. Therefore, the total number of arbitrary constants $2(k_1+k_2+k_3)+k_4+k_5-\alpha'-2\alpha''$ equals the number of additional parameters of the problem.

The solution obtained can be used to examine another problem. We introduce the parameters $\mu_j^*={}^1/_4\mu_j^{-1}$, $q_j^*=q_j^{-1}$, $q_{jk}^*=q_{jk}q_j^{-1}$. Then the right sides of the equalities (5.2) retain their form, on the left side the function u_j' is replaced by $-\sigma_{yj}$ and v_j' by τ_{xyj} and conversely

$$\begin{aligned}&-\mu_j^*\sigma_{yj}=-\mathrm{Re}\,[\varphi_{j1}(z_{j1})+q_{j2}^*\varphi_{j2}(z_{j2})],\\&\mu_j^*\tau_{xyj}=\mathrm{Im}\,[q_{j1}^*\varphi_{j1}(z_{j1})+\varphi_{j2}(z_{j2})]\\&-u_j'=2\,\mathrm{Re}\,[q_j^*\varphi_{j1}(z_{j1})+q_{j2}^*\varphi_{j2}(z_{j2})],\\&v_j'=2\,\mathrm{Im}\,[q_{j1}^*\varphi_{j1}(z_{j1})+q_j^*\varphi_{j2}(z_{j2})]\end{aligned}\tag{5.5}$$

It hence follows that by an appropriate replacement of all six boundary conditions

$$\begin{aligned}&[-u'(x)]=u^\circ(x),\quad [v'(x)]=v^\circ(x),\quad [-\sigma_y(x)]=\sigma^\circ(x),\\&[\tau_{xy}(x)]=\tau^\circ(x),\quad x\in L_1\\&v_j^\circ(x)=v_j^\circ(x),\quad \tau_{xyj}(x)=\tau_j^\circ(x),\quad x\in L_2;\\&-u_j'(x)=u_j^\circ(x),\quad -\sigma_{yj}(x)=\sigma_j^\circ(x),\quad x\in L_3\end{aligned}\tag{5.6}$$

etc. and by replacing the quantities $\mu_j,\ q_j,\ q_{jk},\ u_j^\circ,\ v_j^\circ,\ \sigma_j^\circ,\tau_j^\circ$ by $\mu_j^*,\ q_j^*,\ q_{jk}^*,\ \sigma_j^\circ,\ \tau_j^\circ,\ u_j^\circ,\ v_j^\circ$ in (5.3) and (5.4), these formulas and the representations (5.5) will determine the solution of problem (5.6) for a composite plane with new kinds of boundary conditions.

If the plane is homogeneous, then the conditions of total adhesion of the slot edges to the stamps on arbitrary sections L_7 can be set together with (5.1). The solution of this problem will be the sum of solutions of the two problems (3.1) obtained after partitioning of the conditions $u_j'(x)=u_j^\circ(x)$, $v_j'(x)=v_j^\circ(x)$, $x\in L_7$ and (5.1) into symmetric and skew-symmetric.

6. Wedging of a composite plane. The solution of the Hilbert-Riemann problem (and corresponding problems of elasticity theory) for the case of domains L and M^1 containing semi-infinite intervals differs slightly from the solution of (1.9) and (1.3). It is merely necessary to omit those from the quantities $(z-a_1)$, $(z-b_R)$, $(z-s_1)$, $(z-t_Q)$ in which $a_1=-\infty$, $b_R=\infty$ or $s_1=-\infty$ or $a_1=-\infty$, $t_Q=\infty$ in the latter and to use the method from Sect.1 by classifying the infinitely remote point as a common point of two semi-infinite intervals. Without studying these modifications separately, we will confine ourselves to examining a problem whose approximate solution is the content of /3/.

Let us composite plane be weakened by a semi-infinite slit $-\infty<x<l$, $y=0$, $l>0$, which propagates under the action of a finite stamp $-b\leqslant x\leqslant -a<0$ of constant thickness $2H_1$ and a semi-infinite stamp $-\infty<x<-d$ of thickness $2H_2$ imbedded therein and moving at a sub-Rayleigh velocity c; there is no contact friction and a transverse force Y is applied to the first stamp. Since the half-plane materials are different, the crack edges join at a certain section $[0,l)$. Find the solution in which the juncture at the points $x=0$ and $x=-d$ is smooth, i.e., the stresses at these points are bounded.

The boundary conditions of the problem have the form (5.1) where $\sigma^\circ(x)=\tau^\circ(x)=u^\circ(x)=v^\circ(x)\equiv 0$, $x\in L_1=[l,\infty)$, $\tau_j^\circ(x)=v_j^\circ(x)\equiv 0$, $x\in L_2=(-\infty,-d]\cup[-b,-a]$, $\tau_j^\circ(x)=v^\circ(x)=\sigma^\circ(x)\equiv 0$, $x\in L_4=[0,l)$, $\sigma_j^\circ(x)=\tau_j^\circ(x)\equiv 0$, $x\in L_6=(-a,0)\cup(-d,-b)$, and there are no sections L_3 and L_5.

Taking the solution in the form of (5.2) and (5.3), we obtain a homogeneous combined problem $F_1^+(x)=G(x)F_1^-(x)$, $x\in M^1=L_1$, $\mathrm{Im}\,F_1^+(x)=0$, $x\in L^1=L_2\cup L_4$ and a Dirichlet problem $\mathrm{Im}\,F_2^{\pm}(x)=0$, $x\in[-b,-a]$.

We write down the canonical solution of the first problem

$$\begin{aligned}&X(z)=Z(z)\,e^{i\psi(z)}(z+d)^{-\alpha_1}(z+a)^{-\alpha_2}(z-l)^{-\alpha_3}(z-c_1)^{-1}(z-c_2)^{-1}\\&Z(z)=(z-l)^{-1/2+i\gamma},\quad -\pi<\arg(z+d)\leqslant\pi,\quad 0\leqslant\arg(z-t)\leqslant 2\pi,\\&t=-a,-b,l,c_1,c_2\end{aligned}\tag{6.1}$$

Converting (1.3), as in /9/, we represent the function $\psi(z)$ in the form

$$\psi(z)=\pi\,({}^1/_2+\alpha_1+\alpha_2+\alpha_3)+\varphi_0(z)+\varphi(z)\tag{6.2}$$

$$\varphi_0(z)=\frac{Y(z)}{\pi i}\sum_{k=1}^{3}\int_{L_k^1}\frac{{}^1/_2\pi w_k^+ +\arg(t-c_2)+\arg(t-c_2)}{Y^+(t)(t-z)}\,dt\tag{6.3}$$

$$\varphi(z) = -\gamma Y(z) \int_l^{\infty} \frac{dt}{Y(t)(t-z)}, \quad Y(z) = \sqrt{(z+d)(z+b)(z+a)z(z-l)}$$

$$L_1^1 = (-\infty, -d], \quad L_2^1 = [-b, -a], \quad L_3^1 = [0, l]$$

We consider the behaviour of the function $X(z)$ at the nodes. According to a general rule $\varepsilon_1 = 0, \delta_1 = -{}^1/_2, \varepsilon_2 = \delta_2 = 0, \varepsilon_3 = -{}^1/_2, \delta_3 = 0$. Since $L^* = [-b, -a]$, then

$$\nu_1 = \nu_3 = 0, \quad \lambda_1 = \lambda_2 = \lambda_3 = \nu_2 = -{}^1/_2 \tag{6.4}$$

The function $X(z)$ has root singularities for $z = -d, -b, -a, 0$ and is bounded for $z = l$. In view of the absence of points of discontinuity d_{kl} we obtain $m^{\pm}(z) \equiv 0, N_k = \Delta_k = 0$ and by virtue of (1.6) and (6.4) this means $\alpha_1 = \alpha_3 = 0, \alpha_2 = 1$.

Since $\arg Z^+(x) = \arg Z^-(x) = -{}^1/_2\pi + \gamma \ln(l-x)$ for $x < l$ then according to (1.5) $\omega_k = \theta_k$ or $\omega_1 = \omega_2 = 0, \omega_3 = 1$. Now the numbers $w_k^- = 2(\delta_k + \omega_k - \lambda_k)$ can be found according to the formula $w_1^- = -1, w_2^- = 1, w_3^- = 1$. The desired canonical solution of the problem has the form

$$X(z) = -\frac{i \exp[i\varphi(z) + i\varphi_0(z)]}{(z-c_1)(z-c_2)\sqrt{z(z+a)(z+b)(z+d)}} \tag{6.5}$$

where the functions $\varphi(z), \varphi_0(z), Y(z)$ are expressed by (6.3).

The integers w_k^+ and the complex numbers $c_1, c_2,$ are determined, according to (1.4) and (6.3), by the system of equations $(n = 0, 1)$

$$\sum_{k=1}^{3} \int_{L_k^1} \frac{[{}^1/_2\pi w_k^+ + \arg(t-c_1) + \arg(t-c_2)]\, t^n\, dt}{i\pi Y^+(t)} + \gamma \int_l^{\infty} \frac{t^n\, dt}{Y(t)} = 0$$

The general solution of the combined Dirichlet-Riemann boundary value problem has the form (1.9) and (6.5), where $\Phi(z) = F_1(z)$, $g(t) = f_2^{\pm}(t) \equiv 0$, $\Phi_1(z) \equiv 0$, $r = 4$, $s = 2$, $Y_0(z) = (z-l)^{-{}^1/_2}[z(z+a)(z+b)(z+d)]^{{}^1/_2}$, $Q_s(z) = C_1 z + C_0$. Satisfying the condition of boundedness of the function $F_1(z) = X(z)\Phi_2(z)$ at once for $z = -d, 0$ and consequently setting $P_{r-1}(z) = z(z+d)(D_1 z + D_0)$, we write the solution in the form

$$F_1(z) = (z-c_1)^{-1}(z-c_2)^{-1}\Psi(z) \exp[i\varphi(z) + i\varphi_0(z)]$$

$$\Psi(z) = \frac{C_1 z + C_0}{\sqrt{z-l}} + i(D_1 z + D_0)\sqrt{\frac{z(z+d)}{(z+a)(z+b)}}$$

where C_0, C_1, D_0, D_1 are real constants.

We have two complex equations $\Psi(c_k) = 0, k = 1, 2$ to eliminate the poles of the function $F_1(z)$ at the points $z = c_k$. Finally, solving the homogeneous Dirichlet problem for the function $F_2(z)$ we obtain /6/

$$F_2(z) = iC(z+a)^{-{}^1/_2}(z+b)^{-{}^1/_2} \tag{6.6}$$

The coordinates of the boundary points of the contact sections are determined from the relationships

$$\int_{-d}^{-b} [v'(x, 0)]\, dx = H_1 - H_2, \quad \int_{-a}^{0} [v'(x, 0)]\, dx = -H_1$$

where according to (5.2), (5.3), and (6.6), we have

$$\mu_1 \sqrt{p_2}\, [v'(x)] = G^+ \operatorname{Im} F_1^+(x) - G^- \operatorname{Im} F_1^-(x)$$

The real constant C in (6.6) is found from the given value of the force Y.

Integrating the jump of the normal stresses in $[-b, -a]$, we obtain $C = p_{11}p_{21}(2\pi p_1)^{-1}Y$.

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CONTACT PROBLEMS OF THE MECHANICS OF BODIES WITH ACCRETION*

N.KH. ARUTYUNYAN and A.V. MANZHIROV

Contact problems of the mechanics of bodies with accretion are studied. A general formulation of the mixed problem is given for a viscoelastic ageing body during its continuous piecewise accretion. Complete systems of equations of the mixed problem are given in time intervals from the onset of loading to the onset of accretion, from the onset of accretion to the end of accretion, and beyond it.

The characteristic feature of the basic relations in the case of a body with continuous accretion is the use not of the usual equations of compatibility of the deformations and the Cauchy relations, but of their analogues in the rates of change of the corresponding quantities /1-3/. Moreover, the given previous histories of the deformation tensor of the accruing elements form, at the instant of attachment, specific initial and boundary conditions /2/ on the accruing surface. In particular, the total stress tensor associated with external loads and characterizing the tightness of attachment of the accruing elements is determined at the accruing surface /2, 3/. The instant of attachment of the new elements to the main body represents an important characteristic of the process. The set of instants of attachment completely determines the configuration of the accruing body at any instant of time. Equations of state of the theory of creep of the inhomogeneously ageing bodies are used /4, 5/. The equations reflect the fundamental specific features of the accretion process where the times of preparation and onset of loading play an important part.

A method of solving the mixed and initial-boundary value problems is given. Contact problems for a wedge under various methods of accretion are considered. Integral equations are derived and their solutions constructed. Numerical solutions of the contact problems for a wedge with accretion are given for the case when the influx of matter from outside results in increasing the wedge angle, and for an accruing quarter-plane. Qualitative and quantitative effects are discussed, especially the influence of the method and rate of accretion on the contact characteristics.

1. Formulation and solution of the mixed problem for an ageing, viscoelastic body with accretion. Let a homogeneous, viscoelastic ageing body manufactured at the instant $t = 0$. occupy the region Ω_0 with surface S_0, and be stress-free up to the instant

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